Definition:- A sequence $\langle x_n \rangle$ is said to be converges to l (a finite number) iff for a given $\epsilon > 0$, $\exists$ n_0 (depends on ϵ) $\in N$ s.t.

$$|x_n - l| < \epsilon \quad \forall\ n > n_0$$

In this case $\langle x_n \rangle$ is said to be convergent.
If no finite number l exists, then it is said to be divergent. i.e. either $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$.

Definition:- A sequence $\langle x_n \rangle$ is said to be divergent iff for every K (however large) $\exists$ a natural number $n_0 \in N$ s.t.

$$x_n > K \quad \text{or} < -K \quad \forall\ n > n_0.$$

Definition:- A sequence which is neighter convergent nor divergent, ~~then~~ is called ~~oscillatory~~ oscillatory sequence.

Ex. 1. Sequence $\langle \frac{1}{n^2} \rangle$ converges to 0.

$$\because \lim_{n\to\infty} \left(\frac{1}{n^2}\right) = 0$$

Ex 2. Let $\langle x_n \rangle = \langle 3^n \rangle$, then $\langle x_n \rangle$ is divergent.

$\because$ Let us take a positive real number $K > 0$.

Take $3^n > K \Rightarrow n \log 3 > \log K$

$$\Rightarrow n > \left(\frac{\log K}{\log 3}\right)$$

$\Rightarrow$ If choose the stage $n_0 \geq \left(\frac{\log K}{\log 3}\right)$

then, we can say, that for every K,

$\exists \; n_0 \geq \left(\frac{\log K}{\log 3}\right) \in N$ ~~[illegible]~~ then for $\langle x_n \rangle = \langle 3^n \rangle$

~~$\langle x_n \rangle = 3^n$~~ $\exists \; n_0 \geq \left(\frac{\log(K)}{\log(3)}\right)$

St. $\quad 3^n > K \quad$ ~~$\forall n_0 \in N$~~ $\forall \; n > n_0$.

$\Rightarrow \langle x_n \rangle$ is divergent.

Ex (3). $\langle x_n \rangle = \langle (-1)^n \rangle$, $n \in N$

Here $\langle x_n \rangle = \{-1, 1, -1, 1, \ldots\}$ is bounded

but not convergent

$\Rightarrow$ finitely oscillatory.

Th. Prove that every convergent sequence is bounded.

Proof:— Let $\langle x_n \rangle$ be a sequence, which converges to a finite limit l.

$\Rightarrow$ for a given $\epsilon > 0$; $\exists \; n_0 \in N$. st.

$|x_n - l| < \epsilon \quad \forall \; n > n_0$. —(i)

choose $\epsilon = 1$ $\Rightarrow |x_n - l| < 1 \quad \forall \; n > n_0$.

Then take $|x_n| = |x_n - l + l|$ ~~$\forall n > n_0$~~

$\leq |x_n - l| + |l|$

$< 1 + |l| \quad \forall n > n_0$ using (i)

$\Rightarrow$ maximum value of $x_n = 1 + |l|$

$\Rightarrow |x_n| < 1 + |l|$ after the stage of n_0.

$\Rightarrow$ upto n_0; we have $x_1, x_2, x_3 \ldots \ldots x_{n_0}$

Let $K = \max\{x_1, x_2, x_3, \ldots x_{n_0}, 1+|l|\}$

$\Rightarrow |x_n| < K \quad \forall n \in N$

$\Rightarrow \langle x_n \rangle$ is bounded. H.P.

Th. If $\langle x_n \rangle$ is a convergent sequence, then prove that it's limit is unique.

Proof:- Let us take contrary & let $\langle x_n \rangle$ has two different limits l_1 & l_2.

Because $l_1 \neq l_2 \Rightarrow \exists\, \delta > 0$ st. $|l_1 - l_2| > \delta$ ----(i)

Now for limit l_1: $\lim\limits_{n\to\infty} x_n = l_1$.

So for a given $\epsilon > 0$, $\exists\, n_1 \in N$ st.

$|x_n - l_1| < \epsilon \quad \forall n > n_1$

take $\epsilon = \delta/2$

$\Rightarrow |x_n - l_1| < \delta/2 \quad \forall n > n_1$ ---(ii)

Again for limit l_2: $\lim\limits_{n\to\infty} x_n = l_2$

So for a given $\epsilon > 0$, $\exists\, n_2 \in N$ st

$|x_n - l_2| < \epsilon \quad \forall n > n_2$

chose $\epsilon = \delta/2$

$\Rightarrow |x_n - l_2| < \delta/2 \quad \forall n > n_2$ -- (iii)

Let $n_0 =$ maximum $\{n_1, n_2\}$

then using (ii) & (iii), we have.

$$|x_n - l_1| < \delta/2 \, , \, |x_n - l_2| < \delta/2 \quad \forall\, n > n_0$$

$$\Rightarrow |l_1 - l_2| = |l_1 - x_n + x_n - l_2|$$

$$\leq |x_n - l_1| + |x_n - l_2|$$

$$< \delta/2 + \delta/2 \qquad \forall\, n > n_0$$

$$\Rightarrow |l_1 - l_2| < \delta \quad \forall\, n > n_0$$

which is contrary to our assumption mentioned under (i).

$\Rightarrow$ our assumption was wrong

$\Rightarrow$ $\lim_{n \to \infty} x_n$ are same, not different

$$\Rightarrow \quad l_1 = l_2 = l \text{ (say)}$$

$$\Rightarrow \quad \lim_{n \to \infty} x_n = l$$

H.P.